MATHEMATICS

THE SMALL INDUCTIVE DIMENSION CAN BE RAISED BY THE ADJUNCTION OF A SINGLE POINT

BY

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Unless otherwise indicated, all spaces are metrizable.

1. INTRODUCTION

In this note we construct a metrizable space P containing a point p such that $\operatorname{ind} P\backslash\{p\}=0$ and yet $\operatorname{ind} P=1$, thus showing that the small inductive dimension of a metrizable space can be raised by the adjunction of a single point. (Without the metrizability condition such an example has already been constructed by Dowker [3, p. 258, exercise C]). So the behavior of the small inductive dimension is in contrast with that of the dimension functions Ind and dim, which, as is well known, cannot be raised by the adjunction of a single point [3, p. 274, exercises B and C].

The construction of our example stems from the proof of the equality of ind and Ind on the class of separable metrizable spaces, as given in [10, p. 178]. We make use of the fact that there exists a metrizable space D with $\operatorname{ind} D=0$ and $\operatorname{Ind} D=1$ [14]. This space is completely metrizable. Hence our main example, and the two related examples, also are completely metrizable.

Following [7, p. 208, example 3.3.3] the example is used to show that there is no finite closed sum theorem for the small inductive dimension, even on the class of metrizable spaces. (A dimension function d is said to satisfy the finite closed sum theorem if $d(X \cup Y)=\max\{dX, dY\}$ whenever X and Y are closed).

If f is a continuous closed mapping from X onto Y (X and Y metrizable) such that $f^{-1}(y)$ consists of exactly k points for each $y \in Y$, where k is some fixed integer, then by a theorem of Suzuki $\operatorname{Ind} X=\operatorname{Ind} Y$ [12, p. 73]. As is shown in section 3, there is no analogue of this theorem for ind. Indeed there is a metrizable space T with $\operatorname{ind} T=0$, which admits a continuous closed mapping f onto P such that $f^{-1}(x)$ consists of exactly two points for each $x \in P$.

The paper is organized as follows. After discussing a preliminary lemma in section 2, we give a detailed exposition of the above mentioned examples in section 3. In section 4 several propositions, which are related to the examples above, are presented. It is shown that the equality of the small and of the large inductive dimension (modulo a class of spaces), is logically equivalent to the so-called point-addition theorem.

I wish to thank J. M. Aarts for his helpful comments.

2. A lemma

The following lemma is the key to the construction of the main example P. It is a modification of [10, p. 138].

Lemma. Let A and B disjoint closed subsets of a space X. Then there exists a space Y, a point $a \in Y$ and a continuous mapping f from X onto Y such that

(1) $f^{-1}(a) = A$.

(2) the restriction of f to $X \backslash A$ is a homeomorphism (onto $Y \backslash \{a\}$).

(3) $f(B)$ is closed in Y.

Proof. Let d be a metric on X, let $\phi : X \to [0, 1]$ be a continuous function such that $\phi^{-1}(0) = A$ and $\phi^{-1}(1) = B$. Then $\delta(x, y) = d(x, y) + |\phi(x) - \phi(y)|$ is a metric on X which by the continuity of ϕ is equivalent to d. It is to be noticed that $\delta(A, B) \geqslant 1$.

The underlying set of Y is $(X \backslash A) \cup \{a\}$, where $a \notin X$. A metric ϱ on Y is defined by

$$\varrho(x, y) = \begin{cases} \min \{\delta(x, y), \delta(x, A) + \delta(A, y)\} & \text{for } x \neq a \neq y \\ \delta(A, y) & \text{for } x = a \\ \delta(x, A) & \text{for } y = a \end{cases}$$

We verify that ϱ satisfies the triangle inequality $\varrho(x, y) + \varrho(y, z) \geqslant \varrho(x, z)$ in the two least obvious cases.

(a) $\varrho(x, y) = \delta(x, y)$, $\varrho(y, z) = \delta(y, A) + \delta(A, z)$. Then

$$\varrho(x, y) + \varrho(y, z) = \delta(x, y) + \delta(y, A) + \delta(A, z) \geqslant \delta(x, A) + \delta(A, z) \geqslant \varrho(x, z).$$

(b) $\varrho(x, y) = \delta(x, A) + \delta(A, y)$, $\varrho(y, z) = \delta(y, A) + \delta(A, z)$. Then

$$\varrho(x, y) + \varrho(y, z) = \delta(x, A) + 2\delta(y, A) + \delta(A, z) \geqslant \delta(x, A) + \delta(A, z) \geqslant \varrho(x, z).$$

The function f is defined by

$$f(x) = \begin{cases} a & \text{for } x \in A \\ x & \text{for } x \notin A \end{cases}$$

The function f is continuous since $\varrho(f(x), f(y)) \leqslant \varrho(x, y)$. The function maps sufficiently small δ-balls in $X \backslash A$ isometrically onto ϱ-balls in $Y \backslash \{a\}$, as can easily be seen. Hence f satisfies condition 2 of the lemma. Therefore $f(B)$ is closed in $Y \backslash \{a\}$. Since $\varrho(a, f(B)) \geqslant 1$, $f(B)$ is closed in Y.

3. The examples

Example 1. A metrizable space P containing a point p such that $\operatorname{ind} P \backslash \{p\} = 0$ and yet $\operatorname{ind} P = 1$.

There exists a space D with $\operatorname{ind} D=0$ and $\operatorname{Ind} D=1$ [14]. Let A and B be disjoint closed subsets of D which cannot be separated by the empty set. Let P be a space, p a point of P and f a continuous mapping from D onto P such that $f^{-1}(p)=A$, the restriction of f to $D\backslash A$ is a homeomorphism, and $f(B)$ is closed in P. This is possible in view of the lemma of the preceding section.

Then $\operatorname{ind} P\backslash\{p\}=\operatorname{ind} D\backslash A=0$. But there is no clopen (closed and open) set U in P with $p\in U\subset P\backslash f(B)$, for otherwise $f^{-1}(U)$ is a clopen set in D satisfying $A\subset f^{-1}(U)\subset D\backslash B$, which is impossible. Consequently $\operatorname{ind} P=1$. (See also proposition 5 of section 4).

Remark 1. Similarly the closed subset $f(B)$ of P can be collapsed to a point. This yields a space Q having points a and b which cannot be separated by the empty set, although $\operatorname{ind} Q\backslash\{a, b\}=0$.

Example 2. Failure of the finite closed sum theorem for ind (for metrizable spaces).

The space P of example 1 is the union of the closed subsets F and G defined by

$$F=\{p\}\cup\{y\in P\,|\,2^k\leqslant\varrho(p, y)\leqslant 2^{k+1} \text{ for some odd integer } k\}$$

$$G=\{p\}\cup\{y\in P\,|\,2^k\leqslant\varrho(p, y)\leqslant 2^{k+1} \text{ for some even integer } k\}.$$

Clearly $\operatorname{ind} F=\operatorname{ind} G=0$, but $\operatorname{ind} P=1$. (See also the second part of section 4).

It is known that there is no finite closed sum theorem for ind or for Ind on the class of compact Hausdorff spaces [12, p. 99–102].

Example 3. A space T with $\operatorname{ind} T=0$ and a continuous closed mapping f from T onto the space P of example 1, such that $f^{-1}(x)$ consists of exactly two points for $x\in P$.

Let F and G be as in example 2. We will construct a space K with $\operatorname{ind} K=0$ and a continuous closed mapping ϕ from K onto F such that $\phi^{-1}(x)$ consists of one point for $x\in F\cap G$ and consists of two points for $x\in F\backslash G$. Similarly there exist a space L with $\operatorname{ind} L=0$ and a continuous closed mapping ψ from L onto G such that $\psi^{-1}(x)$ consists of one point for $x\in F\cap G$ and consists of two points for $x\in G\backslash F$.

We may assume that $K\cap L=\emptyset$. Then T is the topological sum of K and L (K and L are closed in T), and f is defined by

$$f(x)=\begin{cases}\phi(x) & \text{for } x\in K\\ \psi(x) & \text{for } x\in L\end{cases}$$

The construction of K is as follows. For convenience $F\cap G$ is denoted by H. The space K will be the double modulo H of F [7, p. 210, definition 3.4.1.]: K is the quotient space $F \times \{0, 1\}/R$, where R is the equivalence relation

$$R = \{((x, i), (x, j)) | i = j \text{ or } x \in H\}.$$

The projection of $F \times \{0, 1\}$ onto K is denoted by π. Since $\pi(F \times \{i\})$ is closed in K (H is closed!) and the restriction of π to $F \times \{i\}$ is one-to-one, this restriction is a homeomorphism, for $i = 0, 1$. Therefore K is the union of two closed metrizable subspaces. So K is metrizable.

Points of $K \backslash \pi(H)$ have arbitrarily small neighborhoods with empty boundary. For $x \in \pi(H)$ the family of clopen sets $\{U \subset K | x \in U, \pi^{-1}(U)$ is clopen in $F \times \{0, 1\}\}$ is a neighborhood base at x. Consequently ind $K = 0$.

The function $\phi: K \to F$ is defined by

$$\phi(x) = y \text{ if } \pi^{-1}(x) = (y, 0) \text{ or } (y, 1).$$

The restriction of ϕ to the closed subspace $\pi(F \times \{i\})$ is a homeomorphism onto F, for $i = 0, 1$. It follows that ϕ is a continuous closed mapping from K onto F. Clearly $\phi^{-1}(x)$ consists of one point for $x \in H = F \cap G$, and consists of two points for $x \in F \backslash G$.

Remark 2. Lokucievskii has given an example of a compact Hausdorff space Z with ind $Z =$ Ind $Z = 2$, which is the union of two closed subspaces Z_0 and Z_1, with ind $Z_i =$ Ind $Z_i = 1$, for $i = 0, 1$ [12, p. 99–102]. The same procedure as above yields a compact Hausdorff space W with ind $W =$ = Ind $W = 1$ and a continuous, necessarily closed mapping f from W onto Z such that $f^{-1}(x)$ consists of two points for each $x \in Z$.

4. Problems related to the sum theorem

In the first part of this section the subbasic dimension is discussed. This new dimension function is of interest in connection with the product theorem. In the second part the relations between sum and addition theorems for the small inductive dimension modulo a class and the equality of the small and large inductive dimension modulo a class are discussed.

We need some terminology. A class $\mathscr{P}$ of spaces is said to be closed monotone if $X \in \mathscr{P}$ whenever X is a closed subspace of a space $Y \in \mathscr{P}$. A class $\mathscr{P}$ is called (closed) additive if $Z \in \mathscr{P}$ whenever Z is the union of two (closed) subspaces which belong to $\mathscr{P}$. Throughout all classes $\mathscr{P}$ are assumed to be topologically closed (i.e. for every $X \in \mathscr{P}$ the class contains all spaces homeomorphic to X) and non-empty.

The boundary operator in X is denoted by B_X and the closure operator in X is denoted by Cl_X. If no confusion is likely to arise the subscript is omitted.

I. The product theorem ind $X \times Y \leqslant$ ind $X +$ ind Y is usually presented as a corollary to the finite closed sum theorem, c.f. [8, p. 33]. It does not hold in general. Indeed, as is shown by a recent example of Filippov, there exist compact Hausdorff spaces X and Y such that ind $X =$ Ind $X = 1$, ind $Y = \operatorname{Ind} Y = 2$ but $\operatorname{ind} X \times Y \geqslant 4$ [4]. It is unknown whether the product theorem for ind holds on the class of metrizable spaces. We introduce the subbasic dimension in relation with this interesting problem.

The subbasic dimension subd is defined by induction as follows. $\operatorname{subd} X = -1$ iff $X = \emptyset$; $\operatorname{subd} X \leqslant n$ if X has a subbase the members of which have boundaries with $\operatorname{subd} \leqslant n-1$, for $n \geqslant 0$; $\operatorname{subd} X = n$ ($n = 0, 1, \ldots, \infty$) are defined as usual.

This dimension function stems from a definition of the small inductive dimension for separable metrizable spaces, as suggested by DE GROOT [6, p. 4].

The subbasic dimension has the following properties

(a) $\operatorname{subd} X \leqslant \operatorname{ind} X$.
(b) $\operatorname{subd} X \leqslant \operatorname{subd} Y$ whenever $X \subset Y$ (c.f. the corresponding property of ind [8, p. 26]).
(c) $\operatorname{subd} X = 0$ iff $\operatorname{ind} X = 0$.
(d) if either subd or ind satisfies the finite closed sum theorem on a closed monotone class $\mathscr{P}$, then subd = ind on $\mathscr{P}$.

Proof of (d); By virtue of (a) it is sufficient to prove by induction on n that $\operatorname{ind} X = \operatorname{subd} X$ if $\operatorname{subd} X = n$. The case $n = 0$ follows from (c).

Suppose that the statement is true for $n \leqslant m$. Let X be a space with $\operatorname{subd} X = m+1$, let $\mathscr{S}$ be a subbase of X the members of which have boundaries with $\operatorname{subd} \leqslant m$. A point of X has arbitrarily small neighborhoods of the form $\cap \mathscr{F}$, where $\mathscr{F}$ is some finite subfamily of $\mathscr{S}$. By the induction hypothesis $\operatorname{subd} B(F) = \operatorname{ind} B(F)$ for $F \in \mathscr{F}$, since $B(F)$ is a closed subset of X and $X \in \mathscr{P}$.

Now $B(\cap \mathscr{F}) \subset \cup \{B(F) | F \in \mathscr{F}\}$, and this union is a member of $\mathscr{P}$ since $\mathscr{P}$ is closed monotone and $X \in \mathscr{P}$. By the sum theorem and (b) we have $\operatorname{ind} B(\cap \mathscr{F}) = \operatorname{subd} B(\cap \mathscr{F}) \leqslant m$. It follows that $m+1 = \operatorname{subd} X \leqslant \operatorname{ind} X \leqslant m+1$, hence $\operatorname{ind} X = \operatorname{subd} X$.

The nicest property of subd is that the product theorem is trivially satisfied.

(e) $\operatorname{subd} X \times Y \leqslant \operatorname{subd} X + \operatorname{subd} Y$.

As is shown by example 2 of section 3, there is no finite closed sum theorem for ind on the class of metrizable spaces. It follows from (d) that subd does not satisfy the finite closed sum theorem. We do not know whether ind = subd on the class of metrizable spaces. Of course, this equality would prove the product theorem for ind on the class of metrizable spaces. As is shown by the aforementioned example of Filippov and (e), subd and ind are not equal on the class of compact Hausdorff spaces.

The class of order totally paracompact metrizable spaces, $\mathscr{T}$, was introduced in [5]. This class includes both the totally paracompact and the strongly paracompact (metrizable) spaces. The class $\mathscr{T}$ is closed monotone, and ind $=$ Ind on $\mathscr{T}$. It follows from the countable closed sum theorem for Ind [12, p. 53 and p. 74] and (d), that subd $=$ ind on $\mathscr{T}$.

II. Let $\mathscr{P}$ be a *nonempty* topologically closed class of spaces. The small (large) inductive dimension modulo $\mathscr{P}$, $\mathscr{P}$-ind ($\mathscr{P}$-Ind) is defined in a similar way as ind (Ind) but starting with the definition that $\mathscr{P}$-ind $X=-1$ ($\mathscr{P}$-Ind $X=-1$) iff $X\in\mathscr{P}$. Observe that $\{\emptyset\}$-ind $=$ ind and $\{\emptyset\}$-Ind $=$ Ind.

LELEK was the first to define such inductive invariants [11]. For information on $\mathscr{P}$-ind the reader is referred to NISHIURA [13]. For information on $\mathscr{P}$-Ind (and also $\mathscr{P}$-dim, the covering dimension modulo $\mathscr{P}$) the reader is referred to the survey paper [2] of AARTS.

The following proposition has a straightforward inductive proof [2, p. 195], [13, p. 247].

PROPOSITION 1. The following conditions on a class $\mathscr{P}$ are equivalent.

(1) $\mathscr{P}$ is closed monotone.
(2) $\mathscr{P}$-ind $X\leqslant\mathscr{P}$-ind Y whenever X is a closed subspace of Y.
(3) $\mathscr{P}$-Ind $X\leqslant\mathscr{P}$-Ind Y whenever X is a closed subspace of Y.

We are interested in sum and addition theorems for $\mathscr{P}$-ind. To be specific, we consider the following three properties of $\mathscr{P}$-ind.

A. The addition theorem: $\mathscr{P}$-ind $X\cup Y\leqslant\mathscr{P}$-ind $X+\mathscr{P}$-ind $Y+1$.
P. The point-addition theorem: $\mathscr{P}$-ind $X\cup\{x\}\leqslant\mathscr{P}$-ind X provided $X\notin\mathscr{P}$. (Observe that $\mathscr{P}$-ind $X\leqslant\mathscr{P}$-ind Y provided that X is an open subspace of Y and $Y\notin\mathscr{P}$. Therefore equality holds if $X\cup\{x\}\notin\mathscr{P}$).
S. The (finite closed) sum theorem: $\mathscr{P}$-ind $X\cup Y=\max\{\mathscr{P}$-ind X, $\mathscr{P}$-ind $Y\}$ whenever X and Y are closed in $X\cup Y$.

Clearly $\mathscr{P}$ is additive if the addition theorem for $\mathscr{P}$-ind holds, and $\mathscr{P}$ is closed monotone and closed additive if the sum theorem for $\mathscr{P}$-ind holds. There is no converse, see remark 1 below. The next three propositions indicate how A, P and S are related.

PROPOSITION 2. The sum theorem for $\mathscr{P}$-ind implies the point-addition theorem for $\mathscr{P}$-ind.

PROOF. See example 2 of section 3. Observe that $\emptyset\in\mathscr{P}$ since $\mathscr{P}$ is closed monotone, in view of our convention that $\mathscr{P}$ be nonempty.

PROPOSITION 3. If $\mathscr{P}$ contains the class of one-point spaces (e.g. if $\mathscr{P}$ is closed monotone and $\mathscr{P}\neq\{\emptyset\}$), then the addition theorem for $\mathscr{P}$-ind implies the point-addition theorem for $\mathscr{P}$-ind.

The proof directly follows from the definitions. Observe that the condition that $\mathscr{P}$ contains the class of one-point spaces is essential: $\{\emptyset\}$-ind satisfies the addition theorem [8, p. 28], but fails to satisfy the point-addition theorem, as our main example shows.

A partial converse of the preceding proposition is given in the following proposition, the proof of which is a minor modification of the proof of the addition theorem for ind [8, p. 28].

PROPOSITION 4. If $\mathscr{P}$ is an additive and closed monotone class, then the point-addition theorem for $\mathscr{P}$-ind implies the addition theorem for $\mathscr{P}$-ind.

PROOF. The proof is by induction on $n = \mathscr{P}\text{-ind}\, X + \mathscr{P}\text{-ind}\, Y + 1$. The additivity of $\mathscr{P}$ settles the case $n = -1$. Assume that the addition theorem is true for $n \leqslant m$ and let $n = m+1$. $\mathscr{P}$ is additive and $n \geqslant 0$. Hence without loss of generality we may assume $X \notin \mathscr{P}$.

Observe that if $\mathscr{Q}$ is a closed monotone class, and $Q \notin \mathscr{Q}$ is a subspace of P, then each point of Q has arbitrarily small neighborhoods U in P such that $\mathscr{Q}\text{-ind}\, Q \cap B_P(U) < \mathscr{Q}\text{-ind}\, Q$, c.f. [8, p. 27].

So if x is any point of $X \cup Y = Z$, x has arbitrarily small neighborhoods U in Z such that

$$\mathscr{P}\text{-ind}\,(X \cup \{x\}) \cap B_Z(U) \leqslant \mathscr{P}\text{-ind}\, X \cup \{x\} - 1 \leqslant \mathscr{P}\text{-ind}\, X - 1.$$

Since $\mathscr{P}$ is closed monotone, $\mathscr{P}\text{-ind}\, Y \cap B_Z(U) \leqslant \mathscr{P}\text{-ind}\, Y$ by proposition 1. Therefore $\mathscr{P}\text{-ind}\, B_Z(U) \leqslant m$ by the induction hypothesis. It follows that $\mathscr{P}\text{-ind}\, X \cup Y \leqslant m+1$.

Proposition 2 was implicity used in [7, p. 208, example 3.3.3] to show that there is no sum theorem for $\mathscr{K}$-ind, where $\mathscr{K}$ is the class of compact spaces. Indeed there is no point addition theorem for $\mathscr{K}$-ind, as the following example shows. If X is the open unit disc in the plane, and x is a boundary point, then $\mathscr{K}\text{-ind}\, X = 0$ but $\mathscr{K}\text{-ind}\, X \cup \{x\} = 1$ [7, p. 207, example 3.3.1].

REMARK 1. The class $\mathscr{K}$ is closed monotone and additive. Nevertheless there is no addition or sum theorem for $\mathscr{K}$-ind.

If $\mathscr{C}$ is the class of completely metrizable spaces, $\mathscr{C}$-Ind satisfies the sum theorem, and $\mathscr{C}\text{-ind} = \mathscr{C}\text{-Ind}$ on the class of separable metrizable spaces [1, p. 35 and p. 39]. The problems whether $\mathscr{C}\text{-ind} = \mathscr{C}\text{-Ind}$ in general, or whether $\mathscr{C}$-ind satisfies the sum theorem, the addition theorem or the point-addition theorem are still open. (As has been pointed out by J. M. Aarts, the statement in [1, p. 39] that the addition theorem for $\mathscr{C}$-ind can be proved directly, is erroneous). In fact these problems are equivalent, as propositions 2, 3 and 4, the sum theorem for $\mathscr{C}$-Ind and the following proposition show.

PROPOSITION 5. Let $\mathscr{P}$ be any class of spaces. If $\mathscr{P}\text{-ind} = \mathscr{P}\text{-Ind}$ then $\mathscr{P}$-ind satisfies the point-addition theorem. The converse is true provided that $\mathscr{P}$ is closed monotone.

PROOF. Assume $\mathscr{P}$-ind$=\mathscr{P}$-Ind. Let X be a space not in $\mathscr{P}$. Assume $x \notin X$, and denote $X \cup \{x\}$ by Y. It is sufficient to prove that x has arbitrarily small neighborhoods U in Y such that $\mathscr{P}$-ind $B_Y(U) < \mathscr{P}$-ind $X = \mathscr{P}$-Ind X.

Let V and W be neighborhoods of x in Y such that $\mathrm{Cl}_Y V \subset W$. Then there is an open subset H of X such that $\mathrm{Cl}_Y V \backslash \{x\} \subset H \subset W$ and $\mathscr{P}$-ind $B_X(H) < \mathscr{P}$-ind X. Then $U = H \cup \{x\}$ is a neighborhood of x in Y and $B_Y(U) = B_X(H)$. It follows that $\mathscr{P}$-ind $B_Y(U) < \mathscr{P}$-ind X.

The proof of the converse essentially is the same as the construction of the main example, cf. [10, p. 178]. First note that the inequality $\mathscr{P}$-ind $\leqslant \mathscr{P}$-Ind has an easy direct proof. We prove that $\mathscr{P}$-Ind $X \leqslant n$ if $\mathscr{P}$-ind $X \leqslant n$ by induction. The case $n = -1$ follows from the definitions. Assume the statement to be correct for $n \leqslant m$. Let X be a space satisfying $0 \leqslant \mathscr{P}$-ind $X \leqslant m+1$. Let A and B be disjoint closed subsets of X.

By the lemma of section 2, there is a space Y, a point $a \in Y$ and a continuous mapping f from X onto Y such that $f^{-1}(a) = A$, the restriction of f to $X \backslash A$ is a homeomorphism and $f(B)$ is closed in Y.

We show that there exists an open neighborhood U of a in Y such that $U \subset Y \backslash f(B)$ and $\mathscr{P}$-ind $B_Y(U) \leqslant m$. If $Y \backslash \{a\} \in \mathscr{P}$, then $U = Y \backslash f(B)$ is the required neighborhood, for $\mathscr{P}$-ind $B_Y(U) = -1$ by the closed monotonicity of $\mathscr{P}$. If on the other hand $Y \backslash \{a\} \notin \mathscr{P}$, then the existence of U follows from the point-addition theorem and the inequality $\mathscr{P}$-ind $Y \backslash \{a\} \leqslant \mathscr{P}$-ind X, which is true since $Y \backslash \{a\}$ is homeomorphic to the open subspace $X \backslash A$ of X, and $X \notin \mathscr{P}$.

The proof is finished as follows. The spaces $B_X(f^{-1}(U))$ and $B_Y(U)$ are homeomorphic, $f^{-1}(U)$ is open and $A \subset f^{-1}(U) \subset X \backslash B$. It follows from the induction hypothesis that $\mathscr{P}$-Ind $B_X(f^{-1}(U)) = \mathscr{P}$-Ind $B_Y(U) \leqslant m$, consequently $\mathscr{P}$-Ind $X \leqslant m+1$.

REMARK 2. This proposition should be compared with the fact that a normal (Hausdorff) space has Ind $X = 0$ iff for every normal space Y, obtained from X by the adjunction of a single point, the equality ind $Y = 0$ holds [9, p. 101].

ADDED IN PRINT: T. Przymusiński has independently constructed essentially the same example as our example 2, in "A note on dimension theory of metric spaces", to appear Fund. Math.

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